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A sharp transition between a trivial 1D BTW model and self-organized critical rice-pile model

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Abstract. A one-dimensional model of a rice-pile is numerically studied for different driving mechanisms. We found that for a sufficiently large system, there is a sharp transition between the trivial behaviour of a 1D BTW model and self-organized critical (SOC) behaviour. Depending on the driving mechanism, the self-organized critical rice-pile model belongs to two different universality classes.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 64.60.Ht Dynamic critical phenomena – 05.70.Ln Nonequilibrium and irreversible thermodynamics

1 Introduction

In their pioneering work [1] Bak, Tang and Wiensenfeld introduced the concept of self-organized criticality (SOC) to describe the behaviour of extended dissipative dynamical systems. The paradigm of SOC is an idealised sandpile where grains added to a pile dissipate their potential energy through avalanches with no characteristic scale [1–10]. Besides the numerical simulations, many different methods were also used to treat the SOC problems. Dynamical mean-field theory [11] gives a unified description of some stochastic SOC systems including the BTW sandpile model and the forest fire model [12]. Langevin-type approaches [13] have been used on a phenomenological basis. Furthermore, a real space renormalization group method [14] provided good estimates of the exponents. Early experimental studies of real sandpiles led to clear disagreement with the numerical models: bounded distributions of avalanche sizes were observed instead of the expected power-law behaviour [15–19]. Using grains of rice, Frette *et al.* [20] showed that the dynamics exhibit self-organized critical behaviour in one case (for grains with a large aspect ratio) but not in another (for less elongated grains). To take into account the changes in the local slopes observed in the rice-pile experiment, Christensen *et al.* proposed a rice-pile model, hereafter called Oslo model [21–23], where the critical slope for each site is a dynamical variable. Here we propose a model for a pile of granular material where we introduce randomness in the relaxation rule and use two types of driving mechanisms: fixed-position driving where the grains are added on the top of the pile, and random-position driving where the grains are added at randomly chosen sites. On the one hand, this model permits the investigation of the transition between the 1D BTW model and the rice-pile model, and on the other hand the study of the effect of the driving mechanisms on the size and transit-time avalanche exponents.

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2 Model

To take into account the changes in the local slopes in the rice pile experiment, Christensen *et al.* [21] proposed a rice-pile model where the critical slope for each site is a dynamical variable. The rice-pile model is based on a linear array of cells labelled by i, $i = 1, 2, ..., L$, with integer variable $h(i)$ assigned to each of them, with a wall at $i = 0$ and an open boundary at $i = L + 1$. Here $h(i)$ is called the local height of the rice-pile at site i. Its local slope is defined as $z(i) = h(i) - h(i + 1)$ for $i = 1, 2, ..., L$. Initially, the system is empty, *i.e.*, $h(i) = 0\ \forall i$. The profile of the pile evolves through two mechanisms, perturbation and relaxation, with a separation in time scales *i.e.* the rate of deposition is slow enough that any avalanche, triggered by a deposited grain, will have ended before a new grain is added. This constitutes one time step.

At each time step, a grain is added to a column i:

$$h(i) \to h(i) + 1. \qquad (1)$$

With the dropping of rice grains, a rice pile is built up. Whenever there is an active column, *i.e.*, $z(i) > z^{c}(i)$, where $z^{c}(i)$ is a slope threshold, one grain of rice will be transferred from this column to its right neighbour according to the following equation:

$$\begin{aligned} h(i) &\to h(i) - 1 \\ h(i+1) &\to h(i+1) + 1, \end{aligned} \qquad (2)$$

and all the unstable sites topple in parallel. The critical slope $z^c(i)$ of a site i remains unchanged if the site is stable but assumes a new value 1 or 2 every time a rice grain on this site has toppled. This toppling rule is equivalent to taking an annealed randomness in the threshold. The toppling of one or more sites is called an avalanche event, and during the avalanche no grains are added to the pile. The avalanche stops when the system reaches a stable state with $z(i) \leq z^c(i)\ \forall i$. Our motivation is to investigate the transition between the 1D BTW model and the rice-pile model.

We modify the rice-pile model as follows: Whenever, $z(i) \leq z^c(i)\ \forall i$, the pile is stable and there is no diffusion of particles. If, at site i, $z(i) > z^c(i)$ (active site), where $z^c(i)$ takes randomly a value 1 or 2 from a uniform distribution $D(z^c(i)) = 1/2[\delta(z^c(i) - 1) + \delta(z^c(i) - 2)]$, where δ is a Dirac function, then this site topples with a probability which depends on its slope $z(i)$, namely: if $z(i) = 2$ and the site i is an active one, it topples with a probability p_1. Furthermore, if $z(i) > 2$, *i.e.* $z(i) > z^c(i)$ for any value taken by $z^c(i)$, it topples with probability p_2. Notice that if we set $p_1 = 0$ and $p_2 = 1$, the model becomes the BTW model with the critical slope $z^c = 2$, and if we set $p_1 = p_2 = 1$, it is just the rice-pile model or Oslo rice-pile model if the driving mechanism is at the top of the pile. When a grain is dropped on the left-end site of the pile, it may make the site unstable. The site topples and transfers a grain of rice to its right neighbour and so on. And in this way avalanches occur. In this paper, we will take p_2 equal to 1 and $p_1 \leq p_2$ because the higher the slope, the higher is the jump probability. So by varying p_1 from 0 to 1, we can change the model from the 1D BTW sandpile model to the rice pile model in a continuous manner. Each avalanche, due to a single added particle, is characterised by its size, denoted by S and defined as the number of topplings. Furthermore, as in reference [21], each grain is associated with a transit time T, which is defined as the time the particle spent in the pile, defined by $T = T_{\text{out}} - T_{\text{in}}$, where T_{out} and T_{in} denote the output and the input times of the grain; these times are measured in the unit of additions of grains. Thus we measure the size of the relaxation event S, the transit-time T, and their corresponding distributions. In the next section, we will consider two ways of driving mechanisms: one way is to add grains to the top of the pile (site $i = 1$) as it is usually made in the Oslo rice-pile model as well as in the experiment. Another way is to add grains to randomly chosen positions (random driving mechanism). This latter way introduces an external stochasticity in our model. Before going ahead, let us recall some differences between the 1D BTW model and the Oslo rice-pile model. The internal randomness in the critical slopes makes the rice-pile model different from the 1D BTW model. For arbitrary initial conditions, the system reaches a stationary state characterised by power laws. However, in the 1D BTW where the randomness is external and the critical slope $z^c(i)$ is constant, the system reaches a stationary trivial behaviour.

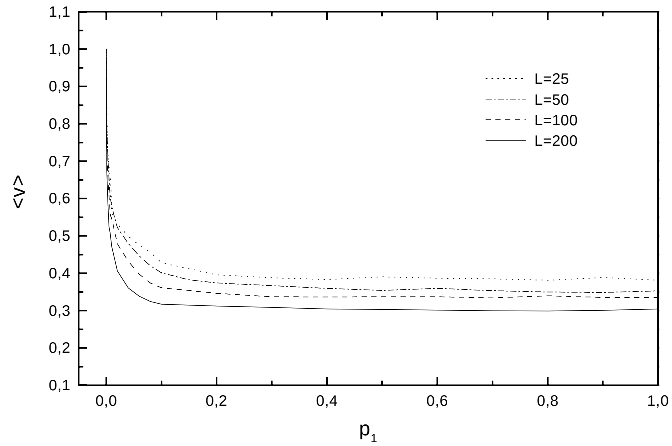

Fig. 1. The average transportation velocity $< v >$, as a function of the probability p_1 for several values of the system size and in the case of a random driving mechanism.

3 Numerical simulations

We have performed extensive numerical simulations and investigated the effects of the parameter p_1 and of driving mechanisms on the behaviour of the system. In the following we will take $p_2 = 1$ and p_1 less than p_2 because the higher the slope, the higher is the jump probability. So by varying p_1 from 0 to 1, we can – in a continuous manner – change the model from the 1D BTW sandpile model to the rice-pile model. Let us first study the transport properties of the model when the external grains are added to randomly chosen positions. In Figure 1, we show the average transportation velocity of grains, defined as $< v >= L/ < T >$, as a function of p_1. For the case $p_1 = 0$ (the 1D BTW model), after a certain transient time a stationary state is reached where every newly-added grain will slip out of the pile instantly, thus the transit time is $T = 0$, and then the average velocity is infinite. By increasing p_1 larger than some value, p_1^c, where p_1^c depends on the system size, the velocity $< v >$ becomes constant, and tends to 0 when L becomes sufficiently large. The numerical results lead us to consider that $p_1^c \to 0$ as $L \to \infty$. For $p_1 = 0^+$, $< v > = 1$, independent of the system size. So there is a sharp transition from $< v > = \infty$ for $p_1 = 0$ to $< v > = 1$ for $p_1 = 0^+$. This transition can be understood by the following argument. It is clear that when p_1 is exactly 0, no newly-added grain will stay in the pile as long the stationary state is reached. So $T = 0$ for every grain and hence $< T > = 0$. When $p_1 = 0^+$ some grains can be buried in the surface layer of the pile. These grains will stay in the pile for a very long time. Once they slip out of the pile, these grains, although very few in number, will make a significant contribution to $< T >$ since their transit times are extremely large. It is the existence of these grains that makes $< T >$ assume a finite value for $p_1 = 0^+$. Between $p_1 = 0^+$ and $p_1 = p_1^c$, there is a crossover behaviour of $< v >$, which is due to finite size effects. Since we expect $p_1^c \to 0$ when $L \to \infty$, we can also expect that for an infinite system the transition takes place

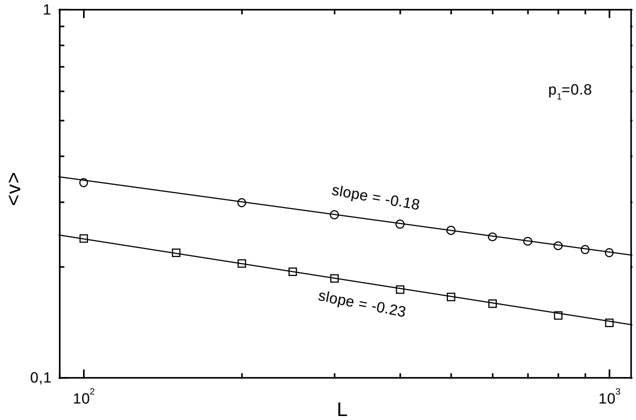

Fig. 2. The average velocity $< v >$ as a function of the system size L, for $p_1 = 0.8$. The open squares correspond to the fixed-position driving. The open circles correspond to the random-position driving.

at $p_1 = 0$ from $< v > = \infty$ to $< v > = 0$. Thus the sharp transition here is induced by tiny disorder. In Figure 2, we plot the average velocity as a function of the system size for $p_1 = 0.8$ and for the two types of driving mechanisms. It is clear that for a large system ($L > 100$), the average velocity $< v >$ scales as $L^{-\gamma}$. When the grains are added on the top of the pile $\gamma = 0.23$ [24], while $\gamma = 0.18$ when the grains are added to randomly chosen sites. Therefore, the average velocity decreases with the system size, which is due to the increase in the active zone depth with system size, as explained by Christensen *et al.* [21]. We have also studied the avalanche size and the transit-time distributions for different values of the probability $p_1 > p_1^c$ and for a random driving mechanism. In Figure 3a we plot our simulation data for $p_1 = 0.8$ and for different system sizes. The distribution is a power law with the presence of a peak close to the cutoff size $S_c \propto L^D$. This is a finite-size effect which is due to the possibility of forming a supercritical state which then relaxes through a very large avalanche. The distribution follows the scaling form:

$$P(S, L) = S^{-\tau} G(S/L^D), \qquad (3)$$

the best collapse is obtained with the exponents $\tau = 1.20 \pm 0.05$ and $D = 1.25 \pm 0.05$, *cf.* Figure 3b. When the grains are added on the top of the pile $\tau = 1.53 \pm 0.05$ and $D = 2.20 \pm 0.05$ [21,24]. Thus by adding an external stochasticity (random driving mechanism) to the internal randomness (critical slope is dynamical variable), the system belongs to another universality class characterised by $\tau = 1.20 \pm 0.05$. By using the fact that the average number of toppling is $< S > = L$ in the critical state, it follows from equation (3) that:

$$\tau = \frac{2D - 1}{D}, \qquad (4)$$

in agreement with our numerical results. Furthermore, the distribution functions of transit times $P(T, L)$ for several values of system sizes are shown in Figure 4. A data collapse for different system sizes L is obtained when plotting

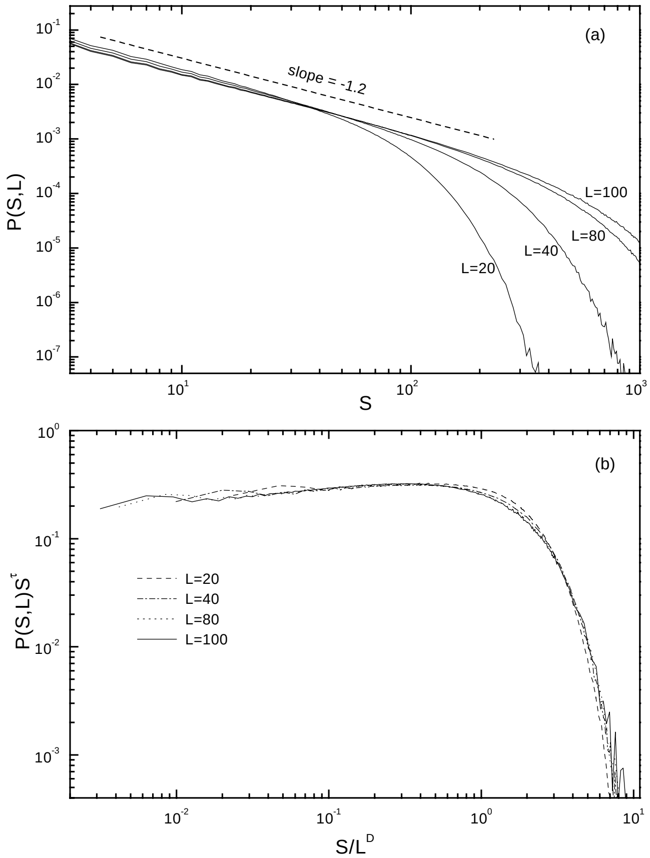

Fig. 3. (a) Log-Log plot of the avalanche size distributions for several values of the system size L with $p_1 = 0.8$ and in the case of the random driving position. (b) Data collapse of the curves displayed in (a) according to equation (3) with the exponents $\tau = 1.20$, $D = 1.25$.

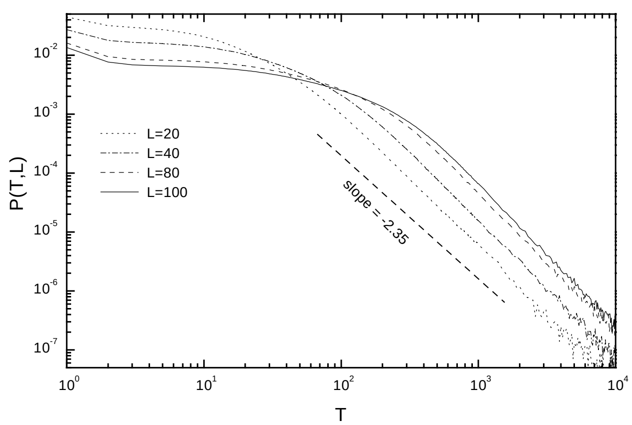

Fig. 4. Log-Log plot of the transit-time distribution for several values of the system size with $p_1 = 0.8$ and in the case of the random driving position.

$L^\beta P(T, L)$ against the rescaled variable T/L^ν

$$P(T, L) = L^{-\beta} F(T/L^\nu), \qquad (5)$$

with $\nu = 1.20 \pm 0.15$ and $\beta = 1.20 \pm 0.15$. The scaling function F is of the form *F(x)* = const. for small x and $F(x) \propto x^{-\alpha}$ for larger x, ν is a critical exponent expressing how the crossover transit time T_c scales with the system

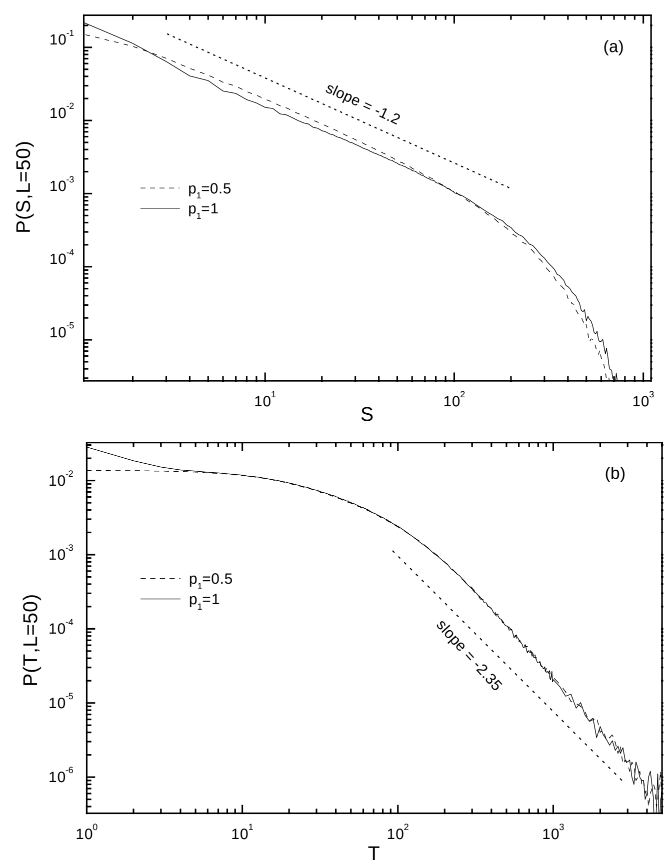

Fig. 5. (a) Log-Log plot of the avalanche size distribution in the case of the random driving position. The best fit gives the slope $\tau = 1.20$. (b) Log-Log plot of the transit time distribution in the case of the random driving position. The best fit gives the slope $\alpha = 2.35$.

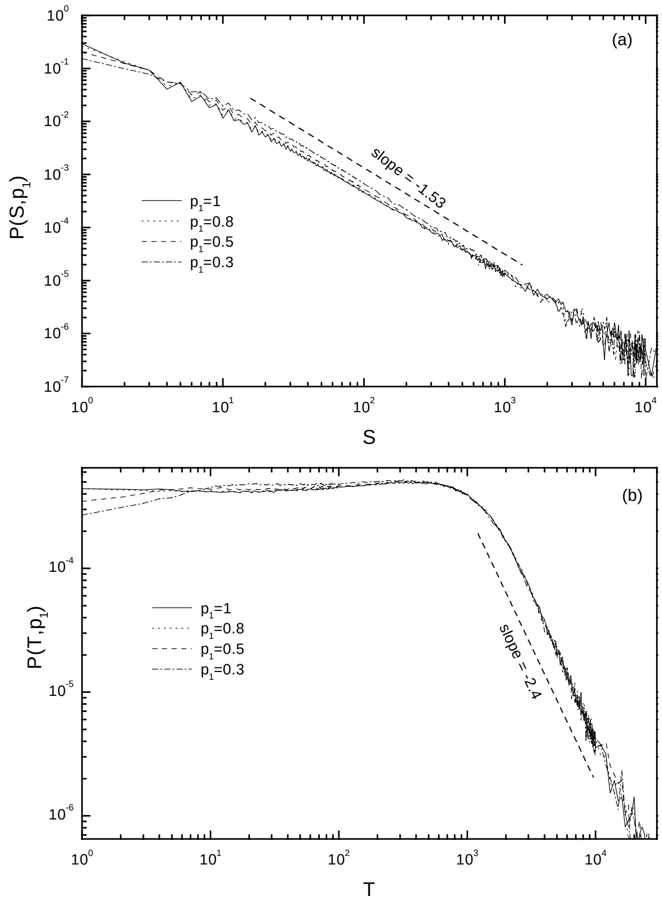

Fig. 6. (a) Log-Log plot of the avalanche size distribution in the case of fixed-position driving. The best fit gives the slope $\tau = 1.53$. (b) Log-Log plot of the transit time distribution in the case of fixed-position driving. The best fit gives the slope $\alpha = 2.40$.

size. The power-law exponent α for large transit time is obtained as $\alpha = 2.35$. Figure 5a shows the avalanche-size distributions for two values of probability p_1 greater than a critical value p_1^c. Figure 5b gives the corresponding transit time distributions. It is clear that the size and the transit time exponents are insensitive to the values of the probability p_1 ($p_1 > p_1^c$). The same conclusion can be drawn in the case where the grains are added on the top of the pile, see Figures 6a and 6b. Thus, the SOC state is insensitive to the variation in the jumping probability $p_1 > 0$. Finally, if we compare the two driving mechanisms, we see clearly that a random external perturbation leads to a decrease of the value of τ because by random driving there is less chance of big avalanches occurring.

4 Conclusion

In summary, we have investigated a one-dimensional rice pile model where the sites with higher slopes have more chance to topple (with a probability $p_2 = 1$) while the sites with lower slopes topple with a probability $p_1 \leq p_2$. It is found that for a sufficiently large system, there is a sharp transition between the trivial behaviour and the SOC behaviour at $p_1 = 0$. In the case where the external grains are added to the top of the pile, the self-organized critical model belongs to the known universality class that is characterised by an avalanche exponent $\tau = 1.53 \pm 0.05$, whereas the model with random driving mechanism belongs to a new universality class characterised by $\tau = 1.20 \pm 0.05$.

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